

ASYMPTOTICALLY LINEAR FRACTIONAL SCHRÖDINGER EQUATIONS

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ABSTRACT. By exploiting a variational technique based upon projecting over the Pohožaev manifold, we prove existence of positive solutions for a class of nonlinear fractional Schrödinger equations having a nonhomogenous nonautonomous asymptotically linear nonlinearity.

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1. INTRODUCTION AND MAIN RESULTS

In the last few years, the study of fractional equations applied to physically relevant situations as well as to many other areas of mathematics has steadily grown. In [21, 22], the authors investigate the description of anomalous diffusion via fractional dynamics and many fractional partial differential equations are derived from Lévy random walk models, extending Brownian walk models in a natural way. In particular, in [17] a fractional Schrödinger equation was obtained, which extends to a Lévy framework a classical result that path integral over Brownian trajectories leads to the standard Schrödinger equation. More precisely, let $s \in (0, 1]$, $n > 2s$ and i be the imaginary unit. Then the Schrödinger equation involving the fractional laplacian $(-\Delta)^s$ is

$$(1.1) \quad i\partial_t u = (-\Delta)^s u - f(x, u), \quad \text{in } (0, \infty) \times \mathbb{R}^n,$$

where the fractional Laplace operator is defined [10], for a suitable constant $C(n, s)$, as

$$(-\Delta)^s u(x) = C(n, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.$$

Though fractional Sobolev spaces are well known since the beginning of the last century, especially among harmonic analysts, they have become very popular in the last few year, under the impulse of the work of Caffarelli and Silvestre [6], see again [10] and the reference within. Looking for standing wave solutions $u(t, x) = e^{i\lambda t} u(x)$ of (1.1) and assuming that the nonlinearity is of the form $f(x, s) = a(x)f(s)$, we are led to study the following fractional equation

$$(1.2) \quad (-\Delta)^s u + \lambda u = a(x)f(u) \quad \text{in } \mathbb{R}^n,$$

for $\lambda > 0$, whose variational formulation (weak solution) is

$$(1.3) \quad \int (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi + \lambda \int u \varphi = \int a(x) f(u) \varphi, \quad \text{for all } \varphi \in H^s(\mathbb{R}^n).$$

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We shall assume that f satisfies the following conditions:

- (f1) $f \in C^1(\mathbb{R}, \mathbb{R}^+)$, $f(s) = 0$ for $s \leq 0$, $\lim_{s \rightarrow 0^+} \frac{f(s)}{s} = 0$;
- (f2) $\lim_{s \rightarrow +\infty} \frac{f(s)}{s} = 1$;
- (f3) if $F(s) := \int_0^s f(t)dt$ and $Q(s) := \frac{1}{2}f(s)s - F(s)$, then there exists $D \geq 1$ such that
$$Q(s) \leq DQ(t), \quad \text{for all } s \in [0, t], \quad \lim_{s \rightarrow +\infty} Q(s) = +\infty.$$

On the function $a : \mathbb{R}^n \rightarrow \mathbb{R}$, we will assume the following conditions:

- (A1) $a \in C^2(\mathbb{R}^n, \mathbb{R}^+)$, $\inf_{\mathbb{R}^n} a > 0$;
- (A2) $\lim_{|x| \rightarrow +\infty} a(x) = a_\infty > \lambda$;
- (A3) $\nabla a(x) \cdot x \geq 0$, for all $x \in \mathbb{R}^n$, with strict inequality on a set of positive measure;
- (A4) $a(x) + \frac{\nabla a(x) \cdot x}{n} < a_\infty$, for all $x \in \mathbb{R}^n$;
- (A5) $\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n} \geq 0$, for all $x \in \mathbb{R}^n$, being \mathcal{H}_a the Hessian matrix of a .

Now we can state our main results. Consider the energy functional $I : H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$,

$$I(u) := \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\lambda}{2} \int u^2 - \int a(x)F(u),$$

naturally associated with equation (1.2). Then, we have the following nonexistence result

Theorem 1.1. *Assume that (A1)-(A5) and (f1)-(f3) hold and consider*

$$\mathcal{P} := \left\{ u \in H^s(\mathbb{R}^n) \setminus \{0\} : \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) - \frac{\lambda}{2} u^2 \right) \right\}.$$

Then, the infimum

$$(1.4) \quad \inf_{u \in \mathcal{P}} I(u),$$

is not a critical level of I and the infimum is not achieved.

Consider now also the limiting problem

$$(1.5) \quad (-\Delta)^s u + \lambda u = a_\infty f(u) \quad \text{in } \mathbb{R}^n.$$

We shall denote by $I_\infty : H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$,

$$I_\infty(u) := \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 + \frac{\lambda}{2} \int u^2 - \int a_\infty F(u),$$

its associated energy functional. In Section 2 we shall discuss some properties of least energy critical values of this functional. In passing, we observe that by combining the results of [20] (see e.g. Theorem 4.1 therein) with an adaptation of [5, (i) of Lemma 1] to the fractional framework, it is possible to prove that *any* least energy solution to (1.5) is radially symmetric and decreasing and of fixed sign.

We have the following existence result

Theorem 1.2. *Assume that (A1)-(A5), (f1)-(f3) hold and that the following facts hold*

- (1) $f \in C^1(\mathbb{R}) \cap \text{Lip}(\mathbb{R}, \mathbb{R}^+)$ and there exists $\tau > 0$ such that $\lim_{s \rightarrow 0^+} \frac{f'(s)}{s^\tau} = 0$;
- (2) $\|a_\infty - a\|_{L^\infty}$ is sufficiently small;

- (3) the least energy level c_∞ of (1.5) is an isolated radial critical level for I_∞ or equation (1.5) admits a unique positive solution which is radially symmetric about some point.

Then the nonautonomous problem

$$(-\Delta)^s u + \lambda u = a(x)f(u) \quad \text{in } \mathbb{R}^n,$$

admits a nontrivial nonnegative solution $u \in H^s(\mathbb{R}^n)$.

These results extend the corresponding results in [18] to the fractional case. The framework employed and ideas of the proofs of our main results follow closely those found in [18]. However, the nonlocal character of the fractional laplacian requires to overcome several additional difficulties.

Theorem 1.2 follows under uniqueness of positive radial solutions of (1.5) or isolatedness assumption on the least energy level of I_∞ . To our knowledge, in the case $s \in (0, 1)$, the isolatedness or uniqueness assumption of Theorem 1.2 are unknown in the current literature. In the case $s = 1$, it follows for instance by the uniqueness result by Serrin-Tang [24], under suitable assumptions of $f(s)$ for large values of s , which are compatible with the model nonlinearity

$$(1.6) \quad f(s) = \frac{s^3}{1+s^2}, \quad \text{for } s \geq 0, \quad f(s) = 0, \quad \text{for } s \leq 0.$$

In fact, assumptions (H1)-(H2) in [24, Theorem 1] are fulfilled with $b = (\lambda/(a_\infty - \lambda))^{1/2} > 0$, where $a_\infty > \lambda$. For the case of superquadratic nonlinearities $f(s) = s^p$, nondegeneracy and uniqueness properties of ground state solutions of (1.5) were recently proved in [13, 14], so assumption (3) of Theorem 1.2 is expected to be fulfilled. Semi-linear Schrödinger equations associated with the asymptotically linear model nonlinearity (1.6) are one of the main motivations for developing the technique in this paper. For the physical background in the local case $s = 1$, see [27, 28].

In [7], the author considers asymptotically linear fractional NLS with an external potential V which provides compactness directly via coercivity. We also refer the reader to the contributions [8, 12] where the case of a superquadratic nonlinearity is covered for the fractional laplacian obtaining existence, regularity and qualitative properties of solutions. In the superquadratic case, as known, one can also exploit the Nehari manifold associated with the problem. On the other hand, when the nonlinear term is nonhomogeneous and asymptotically linear, as it was pointed out by Costa and Tehrani in [9], in general, not every nonzero function can be projected onto the Nehari manifold or it may happen that the projection is not uniquely determined. In turn, as exploited in other contributions [3, 16, 18], we shall look at projections onto the Pohožaev manifold in place of the Nehari constraint in order to prove Theorem 1.1 and 1.2.

A few additional remarks. Conditions (A2), (A3) and (A4) imply

$$(1.7) \quad \nabla a(x) \cdot x \rightarrow 0, \quad \text{if } |x| \rightarrow +\infty,$$

while (f1) and (f2) imply that, given $\varepsilon > 0$ and $2 < p \leq 2n/(n - 2s)$, there exists $C_\varepsilon > 0$ with

$$(1.8) \quad |F(s)| \leq \frac{\varepsilon}{2}|s|^2 + C_\varepsilon|s|^p, \quad \text{for all } s \in \mathbb{R}.$$

In what follows we will denote

$$(1.9) \quad \|u\|_{H^s} = \left(\int |(-\Delta)^{s/2} u|^2 + \int \lambda u^2 \right)^{1/2},$$

as the norm in $H^s(\mathbb{R}^n)$, which is equivalent to the standard norm of $H^s(\mathbb{R}^n)$. We will also denote by $\|u\|_p$ the usual norm of $L^p(\mathbb{R}^n)$. We define $I : H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ as the functional associated with (1.2)

$$I(u) := \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 - \int G(x, u), \quad G(x, u) := a(x)F(u) - \frac{\lambda}{2}u^2.$$

Since $f(s) = 0$ on \mathbb{R}^- , it follows that any weak solution $u \in H^s(\mathbb{R}^n)$ for (1.2) is nonnegative. In fact, by choosing $\varphi = u^- \in H^s(\mathbb{R}^n)$ in the variational formulation (1.3) yields

$$\int (-\Delta)^{s/2} u (-\Delta)^{s/2} u^- = \int a(x) f(u) u^- - \lambda \int u u^- = \lambda \int (u^-)^2.$$

Hence, if $C(n, s)$ is the normalization constant in the definition of $(-\Delta)^s$, we obtain

$$\begin{aligned} (1.10) \quad & \int (-\Delta)^{s/2} u (-\Delta)^{s/2} u^- = \int u^- (-\Delta)^s u^+ - \|(-\Delta)^{s/2} u^-\|_2^2 \\ &= \frac{C(n, s)}{2} \iint \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x - y|^{n+2s}} dx dy - \|(-\Delta)^{s/2} u^-\|_2^2 \\ &= -C(n, s) \iint \frac{u^+(x) u^-(y)}{|x - y|^{n+2s}} dx dy - \|(-\Delta)^{s/2} u^-\|_2^2 \leq -\|(-\Delta)^{s/2} u^-\|_2^2. \end{aligned}$$

In turn we get $\|u^-\|_{H^s}^2 = \|u^-\|_2^2 + \|(-\Delta)^{s/2} u^-\|_2^2 = 0$, namely $u^- = 0$, hence the assertion.

2. ENERGY LEVELS OF THE LIMITING PROBLEM

In this section we study the following equation for $s \in (0, 1)$ and $n > 2s$,

$$(2.1) \quad (-\Delta)^s u + \lambda u = a_\infty f(u), \quad \text{in } \mathbb{R}^n,$$

where $\lambda > 0$ and $a_\infty > \lambda$. We shall assume that F satisfies the growth estimate (1.8). Our aim is to provide a Mountain Pass characterization for least energy solutions which is the counterpart of the main result of [16]. Let the Hilbert space $H^s(\mathbb{R}^n)$ be endowed with the norm (1.9) and let $I_\infty : H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ be the functional corresponding to (2.1), namely

$$I_\infty(u) = \frac{1}{2} \int |(-\Delta)^{s/2} u|^2 - \int G_\infty(u),$$

where we have set

$$G_\infty(u) = \int_0^u g_\infty(t) dt = \int_0^u (a_\infty f(t) - \lambda t) dt = a_\infty F(u) - \frac{\lambda}{2} u^2.$$

We say that a solution u of (2.1) is a *least energy solution* to (2.1) if

$$I_\infty(u) = m, \quad m := \inf \{ I_\infty(u) : u \in H^s(\mathbb{R}^n) \setminus \{0\} \text{ is a solution of (2.1)} \}.$$

As stated in [23, Theorem 1.1], the Pohožaev identity associated with (2.1) is given by

$$(n - 2s) \int u g_\infty(u) = 2n \int G_\infty(u),$$

where g_∞ and G_∞ are defined as before. Also, if u and v belong to $H^s(\mathbb{R}^n)$, then

$$\int v (-\Delta)^s u = \int (-\Delta)^{s/2} u (-\Delta)^{s/2} v,$$

which yields in turn

$$\int u g_\infty(u) = \int |(-\Delta)^{s/2} u|^2 = \frac{C(n, s)}{2} [u]_{H^s}^2, \quad [u]_{H^s} := \left(\iint \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

Therefore, the Pohožaev identity may be written as (see also [8, Proposition 4.1] for a different proof)

$$(2.2) \quad (n - 2s) \int |(-\Delta)^{s/2} u|^2 = 2n \int G_\infty(u).$$

For the following, it is convenient to introduce the set

$$\mathcal{P}_\infty := \{ u \in H^s(\mathbb{R}^n) \setminus \{0\} : u \text{ satisfies identity (2.2)} \}.$$

We also consider we set of paths

$$\Gamma_\infty := \{\gamma \in C([0, 1], H^s(\mathbb{R}^n)) : \gamma(0) = 0, I_\infty(\gamma(1)) < 0\},$$

and define the min-max Mountain Pass level (see [2])

$$(2.3) \quad c_\infty := \min_{\gamma \in \Gamma_\infty} \max_{t \in [0, 1]} I_\infty(\gamma(t)).$$

The main result of the section is the following

Theorem 2.1. $c_\infty = m$.

In order to prove the result we need the following Lemmas.

Lemma 2.2. *Let $w \in H^s(\mathbb{R}^n)$ be a least energy solution to (2.1). Then there exists $\gamma \in \Gamma_\infty$ such that*

$$w \in \gamma([0, 1]), \quad \max_{t \in [0, 1]} I_\infty(\gamma(t)) = I_\infty(w) = m.$$

Proof. Consider a least energy solution w of (2.1), which exists e.g. by [8, Theorem 1.1]. Then we can define the continuous path $\alpha : [0, \infty) \rightarrow H^s(\mathbb{R}^n)$ by setting $\alpha(t)(x) := w(x/t)$, if $t > 0$, and $\alpha(0) := 0$. Then, by construction, we have $I_\infty(\alpha(0)) = 0$ and

$$\begin{aligned} I_\infty(\alpha(t)) &= \frac{1}{2} \int |(-\Delta)^{s/2} w(x/t)|^2 - \int G_\infty(w(x/t)) \\ &= \frac{t^{n-2s}}{2} \int |(-\Delta)^{s/2} w(x)|^2 - t^n \int G_\infty(w), \quad t > 0. \end{aligned}$$

Then, taking the derivative, we obtain

$$\begin{aligned} \frac{d}{dt} I_\infty(\alpha(t)) &= \frac{(n-2s)}{2} t^{n-2s-1} \int |(-\Delta)^{s/2} w|^2 - n t^{n-1} \int G_\infty(w) \\ &= \frac{t^{n-2s-1}}{2} \left\{ (n-2s) \int |(-\Delta)^{s/2} w|^2 - 2n t^{2s} \int G_\infty(w) \right\} \end{aligned}$$

Since w is a solution of (2.1), it satisfies the Pohožaev identity (2.2), therefore

$$\frac{d}{dt} I_\infty(\alpha(t)) = \frac{t^{n-2s-1}}{2} (n-2s)(1-t^{2s}) \int |(-\Delta)^{s/2} w|^2.$$

Then, since $n > 2s$, the map $\{t \mapsto I_\infty(\alpha(t))\}$ achieves the maximum value at $t = 1$. By choosing $L > 0$ sufficiently large and recalling (2.2) again to guarantee $\int G_\infty(w) > 0$, we have

$$\max_{0 \leq t \leq L} I_\infty(\alpha(t)) = I_\infty(\alpha(1)) = I_\infty(w) = m, \quad I_\infty(\alpha(L)) < 0.$$

Taking $\gamma(t) := \alpha(tL)$, we have that $\gamma \in \Gamma_\infty$ and the result follows. \square

Lemma 2.3. $\gamma([0, 1]) \cap \mathcal{P}_\infty \neq \emptyset$, for all $\gamma \in \Gamma_\infty$.

Proof. Consider the functional associated with the Pohožaev identity (2.2),

$$(2.4) \quad J_\infty(u) := \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \int G_\infty(u), \quad u \in H^s(\mathbb{R}^n).$$

We will first prove that there exists $\rho > 0$ such that, if $0 < \|u\|_{H^s} \leq \rho$, then $J(u) > 0$. We have

$$\begin{aligned} J_\infty(u) &= \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n a_\infty \int F(u) + \frac{n\lambda}{2} \int u^2 \\ &\geq \frac{n-2s}{2} \|u\|_{H^s}^2 - n a_\infty \int F(u). \end{aligned}$$

Then, by virtue of (1.8) and the fractional Sobolev inequality [10, Theorem 6.7], we obtain

$$\begin{aligned} J_\infty(u) &\geq \frac{n-2s}{2} \|u\|_{H^s}^2 - \frac{n\varepsilon a_\infty}{2\lambda} \int \lambda u^2 - na_\infty C_\varepsilon \int |u|^p \\ &\geq \frac{1}{2} \left(n - 2s - \frac{n\varepsilon a_\infty}{\lambda} \right) \|u\|_{H^s}^2 - na_\infty C_\varepsilon \|u\|_{H^s}^p. \end{aligned}$$

Take now $\varepsilon > 0$ so small that $n - 2s - n\varepsilon a_\infty/\lambda > 0$ and then choose $\rho > 0$ small enough so that $J_\infty(u) > 0$ if $0 < \|u\|_{H^s} \leq \rho$, which is possible, since $p > 2$. Observe now that

$$J_\infty(u) = nI_\infty(u) - s \int |(-\Delta)^{s/2} u|^2.$$

If $\gamma \in \Gamma_\infty$, we have $J_\infty(\gamma(0)) = 0$ and $J_\infty(\gamma(1)) \leq nI_\infty(\gamma(1)) < 0$. Then, by continuity, there exists $\sigma \in (0, 1)$ such that $\|\gamma(\sigma)\|_{H^s} \geq \rho$ and $J_\infty(\gamma(\sigma)) = 0$. This means $\gamma(\sigma) \in \mathcal{P}_\infty$, concluding the proof. \square

Lemma 2.4. *We have*

$$m = \inf_{u \in \mathcal{P}_\infty} I_\infty(u).$$

Proof. If we set

$$\mathcal{S}_\infty = \left\{ u \in H^s(\mathbb{R}^n) : \int G_\infty(u) = 1 \right\},$$

it follows that $\Phi : \mathcal{S}_\infty \rightarrow \mathcal{P}_\infty$ defined by

$$\Phi(u)(x) := u\left(\frac{x}{t_u}\right), \quad t_u := \left(\frac{n-2s}{2n}\right)^{1/2s} \|(-\Delta)^{s/2} u\|_2^{1/s}$$

establishes a bijective correspondence and

$$I_\infty(\Phi(u)) = \frac{s}{n} \left(\frac{n-2s}{2n}\right)^{(n-2s)/2s} \|(-\Delta)^{s/2} u\|_2^{n/s}, \quad u \in \mathcal{S}_\infty,$$

yielding in turn

$$\inf_{u \in \mathcal{P}_\infty} I_\infty(u) = \inf_{u \in \mathcal{S}_\infty} I_\infty(\Phi(u)) = \inf_{u \in \mathcal{S}_\infty} \frac{s}{n} \left(\frac{n-2s}{2n}\right)^{(n-2s)/2s} \|(-\Delta)^{s/2} u\|_2^{n/s} = m,$$

since the last infimum is achieved and the corresponding value equals the least energy level m . This can be proved by performing calculations similar to those of [5, proof of (i) of Lemma 1]. \square

Proof of Theorem 2.1 concluded. By combining Lemma 2.3 with 2.4 we immediately obtain $m \leq c_\infty$. Considering the path $\gamma \in \Gamma_\infty$ provided by Lemma 2.3, we have

$$\max_{0 \leq t \leq 1} I_\infty(\gamma(t)) = I_\infty(w) = m.$$

By taking the infimum over Γ_∞ yields

$$\inf_{\gamma \in \Gamma_\infty} \max_{t \in [0,1]} I_\infty(\gamma(t)) \leq m,$$

so that $c_\infty \leq m$, which concludes the proof. \square

3. PROJECTING ON THE POHOŽAEV MANIFOLD

From [23, Proposition 1.12], if $u \in H^s(\mathbb{R}^n)$ is a solution of (1.2), then u satisfies de Pohožaev identity

$$(3.1) \quad \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) - \frac{\lambda}{2} u^2 \right).$$

Furthermore, we define the Pohožaev manifold associated with (1.2) by

$$\mathcal{P} := \{u \in H^s(\mathbb{R}^n) \setminus \{0\} : u \text{ satisfies identity (3.1)}\}.$$

We first have the following

Lemma 3.1. *Let the functional $J : H^s(\mathbb{R}^n) \rightarrow \mathbb{R}$ be defined by*

$$J(u) := \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) - \frac{\lambda u^2}{2} \right).$$

Then, it holds that

- a) $\{u \equiv 0\}$ is an isolated point of $J^{-1}(\{0\})$;
- b) $\mathcal{P} := \{u \in H^s(\mathbb{R}^n) \setminus \{0\} : J(u) = 0\}$ is a closed set.
- c) \mathcal{P} is a C^1 manifold.
- d) There exists $\sigma > 0$ such that $\|u\|_{H^s} > \sigma$, for all $u \in \mathcal{P}$.

Proof. (a) Using condition (A4), we get

$$\begin{aligned} J(u) &= \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) - \frac{\lambda u^2}{2} \right) \\ &> \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \int \left(a_\infty F(u) - \lambda \frac{u^2}{2} \right) \\ &\geq \frac{n-2s}{2} \|u\|_{H^s}^2 - na_\infty \int F(u). \end{aligned}$$

By virtue of the fractional Sobolev embedding [10, Theorem 6.7] and (1.8), we obtain

$$\begin{aligned} J(u) &\geq \frac{n-2s}{2} \|u\|_{H^s}^2 - \frac{n\varepsilon a_\infty}{2\lambda} \int \lambda u^2 - na_\infty C_\varepsilon \int |u|^p \\ &\geq \frac{1}{2} \left(n-2s - \frac{n\varepsilon a_\infty}{\lambda} \right) \|u\|_{H^s}^2 - na_\infty C_\varepsilon \|u\|_{H^s}^p. \end{aligned}$$

Take $\varepsilon > 0$ with $n-2s - n\varepsilon a_\infty/\lambda > 0$. For $\rho > 0$ small enough, $J(u) > 0$ if $0 < \|u\|_{H^s} < \rho$.

(b) $J(u)$ is a C^1 functional, thus $\mathcal{P} \cup \{0\} = J^{-1}(\{0\})$ is a closed subset. Moreover, $\{u \equiv 0\}$ is an isolated point in $J^{-1}(\{0\})$ and the assertion follows.

(c) Considering the derivative of J at u and applied at u yields

$$(3.2) \quad J'(u)(u) = (n-2s) \int |(-\Delta)^{s/2} u|^2 - n \int (a(x)f(u)u - \lambda u^2) - \int \nabla a(x) \cdot x f(u)u.$$

Since $u \in \mathcal{P}$, it follows that u satisfies (3.1) and, using formula (3.1) into (3.2), we obtain

$$\begin{aligned}
J'(u)(u) &= 2n \int a(x)F(u) - n\lambda \int u^2 + 2 \int \nabla a(x) \cdot xF(u) \\
&\quad - n \int a(x)f(u)u + n\lambda \int u^2 - \int \nabla a(x) \cdot xf(u)u \\
&= 2n \int \left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) \\
&\quad - n \int \left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) f(u)u \\
&= 2n \int \left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) \left(F(u) - \frac{1}{2}f(u)u \right) < 0,
\end{aligned}$$

in light of (A1), (A3) and (f3). Then, if $u \in \mathcal{P}$, then $J'(u)(u) < 0$. This shows that the set \mathcal{P} is a C^1 manifold.

(d) Since 0 is isolated in $J^{-1}(\{0\})$, there is a ball $\|u\|_{H^s} \leq \sigma$ which does not intersect \mathcal{P} . \square

4. NONEXISTENCE RESULTS

In this section we get relations between the Pohožaev manifold \mathcal{P} associated with (1.2) and the Pohožaev manifold \mathcal{P}_∞ for the limiting problem (2.1). Recall that

$$\mathcal{P}_\infty = \{u \in H^s(\mathbb{R}^n) \setminus \{0\} : J_\infty(u) = 0\},$$

where J_∞ is defined as in (2.4). Notice that the hypotheses (A3)-(A4) imply that $I_\infty(u) < I(u)$ for every u in $H^s(\mathbb{R}^n) \setminus \{0\}$. If p is defined as in (1.4), we will show in this section that $p = c_\infty$, that this level is not critical for I and in turn that it is not achieved.

Lemma 4.1. *If $\int G_\infty(u) > 0$, there exist unique $\vartheta_1, \vartheta_2 > 0$ with $u(\cdot/\vartheta_1) \in \mathcal{P}_\infty$ and $u(\cdot/\vartheta_2) \in \mathcal{P}$.*

Proof. First, we consider the case of \mathcal{P}_∞ . Consider the function $\varphi : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$\varphi(\vartheta) := I_\infty(u(x/\vartheta)) = \frac{\vartheta^{n-2s}}{2} \int |(-\Delta)^{s/2}u|^2 - \vartheta^n \int G_\infty(u).$$

Taking the derivative of φ , we obtain

$$\begin{aligned}
\varphi'(\vartheta) &= \frac{\vartheta^{n-2s-1}}{2} \left((n-2s) \int |(-\Delta)^{s/2}u|^2 - 2n\vartheta^{2s} \int G_\infty(u) \right) \\
&= \frac{1}{\vartheta} \left(\frac{n-2s}{2} \int |(-\Delta)^{s/2}u(x/\vartheta)|^2 - n \int G_\infty(u(x/\vartheta)) \right) = \frac{J_\infty(u(\cdot/\vartheta))}{\vartheta}.
\end{aligned}$$

Then, $\varphi'(\vartheta) = 0$ if and only if either $\vartheta = 0$ or

$$\vartheta = \vartheta_1 = \left(\frac{n-2s}{2n} \frac{\int |(-\Delta)^{s/2}u|^2}{\int G_\infty(u)} \right)^{1/2s} > 0.$$

Since by the formula for φ' we have $u(x/\vartheta) \in \mathcal{P}_\infty$ if and only if $\varphi'(\vartheta) = 0$ for some $\vartheta > 0$, we have the result. In passing, we observe that φ is positive for $\vartheta > 0$ small while it is negative for $\vartheta > 0$ large, so that the unique critical point of φ corresponds to a global maximum point for φ . Now we

turn to the case of \mathcal{P} . First, we define the function $\Psi : (0, \infty) \rightarrow \mathbb{R}$ by

$$\begin{aligned}
 (4.1) \quad \Psi(\vartheta) &:= I(u(x/\vartheta)) = \frac{\vartheta^{n-2s}}{2} \int |(-\Delta)^{s/2} u|^2 - \int G(x, u(x/\vartheta)) \\
 &= \frac{\vartheta^{n-2s}}{2} \int |(-\Delta)^{s/2} u|^2 - \int \left(a(x) F(u(x/\vartheta)) - \lambda \frac{u^2(x/\vartheta)}{2} \right) \\
 &= \frac{\vartheta^{n-2s}}{2} \int |(-\Delta)^{s/2} u|^2 - \vartheta^n \int \left(a(\vartheta x) F(u) - \lambda \frac{u^2}{2} \right).
 \end{aligned}$$

Taking the derivative of Ψ and recalling that $n > 2s$, we obtain:

$$\begin{aligned}
 (4.2) \quad \Psi'(\vartheta) &= \frac{n-2s}{2} \vartheta^{n-2s-1} \int |(-\Delta)^{s/2} u|^2 - n \vartheta^{n-1} \int \left(a(\vartheta x) F(u) - \lambda \frac{u^2}{2} \right) \\
 &\quad - \vartheta^n \int \nabla a(\vartheta x) \cdot x F(u) \\
 &= \vartheta^{n-2s-1} \left\{ \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \vartheta^{2s} \int \left(a(\vartheta x) F(u) - \lambda \frac{u^2}{2} \right) \right. \\
 &\quad \left. - \vartheta^{2s} \int \nabla a(\vartheta x) \cdot (\vartheta x) F(u) \right\} \\
 &= \frac{1}{\vartheta} \left\{ \frac{n-2s}{2} \int |(-\Delta)^{s/2} u(x/\vartheta)|^2 - n \int \left(a(x) F(u(x/\vartheta)) - \lambda \frac{u^2(x/\vartheta)}{2} \right) \right. \\
 &\quad \left. - \int \nabla a(x) \cdot x F(u(x/\vartheta)) \right\} = \frac{J(u(\cdot/\vartheta))}{\vartheta}.
 \end{aligned}$$

Hence, $u(\cdot/\vartheta) \in \mathcal{P}$ if and only if $\Psi'(\vartheta) = 0$, for some $\vartheta > 0$. Notice that, in view of (A2) and (1.7) and the Lebesgue Dominated Convergence Theorem, we get

$$\begin{aligned}
 \lim_{\vartheta \rightarrow \infty} \int a(\vartheta x) F(u) - \lambda \frac{u^2}{2} &= \int G_\infty(u), \\
 \lim_{\vartheta \rightarrow \infty} \int \nabla a(\vartheta x) \cdot (\vartheta x) F(u) &= 0.
 \end{aligned}$$

Therefore, if $\vartheta > 0$ is sufficiently large, then

$$\Psi'(\vartheta) = \vartheta^{n-2s-1} \left\{ \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \vartheta^{2s} \left(\int G_\infty(u) + o_\vartheta(1) \right) \right\}.$$

Since $\int G_\infty(u) > 0$, it follows that $\Psi'(\vartheta) < 0$, for $\vartheta > 0$ sufficiently large. On the other hand, if $\vartheta > 0$ is sufficiently small we have that condition (A4), together with (A1)-(A3) yield

$$\begin{aligned}
 0 &< a(x) + \frac{\nabla a(x) \cdot x}{n} < a_\infty, \\
 -\frac{\lambda}{2} \int u^2 &\leq \int \left(\left(a(\vartheta x) + \frac{\nabla a(\vartheta x) \cdot (\vartheta x)}{n} \right) F(u) - \lambda \frac{u^2}{2} \right) < \int G_\infty(u) \leq \frac{a_\infty C}{2} \int u^2,
 \end{aligned}$$

where C is a positive constant independent of ϑ . Thus, taking $\vartheta > 0$ sufficiently small in $\Psi'(\vartheta)$, we obtain $\Psi'(\vartheta) > 0$. Since Ψ' is continuous, there exists $\vartheta_2 = \vartheta_2(u) > 0$ such that $\Psi'(\vartheta_2) = 0$, which means that $u(\cdot/\vartheta_2) \in \mathcal{P}$. To show the uniqueness of ϑ_2 , note that $\Psi'(\vartheta) = 0$ implies

$$(4.3) \quad \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n \vartheta^{2s} h(\vartheta), \quad h(\vartheta) := \int \left(\left(a(\vartheta x) + \frac{\nabla a(\vartheta x) \cdot (\vartheta x)}{n} \right) F(u) - \frac{\lambda u^2}{2} \right),$$

with $\vartheta > 0$. Taking the derivative of h we end up with

$$h'(\vartheta) = \frac{1}{\vartheta} \int \left(\nabla a(\vartheta x) \cdot (\vartheta x) + \frac{(\vartheta x) \cdot \mathcal{H}_a(\vartheta x) \cdot (\vartheta x)}{n} + \frac{\nabla a(\vartheta x) \cdot (\vartheta x)}{n} \right) F(u).$$

Hypotheses (A3) and (A5) imply that $h'(\vartheta) > 0$. Therefore, h is an increasing function of ϑ and hence there exists a unique $\vartheta > 0$ such that the identity in (4.3) holds. As for the functional φ , the above arguments show that Ψ is positive for $\vartheta > 0$ small while it is negative for $\vartheta > 0$ large, and hence the unique critical point of Ψ corresponds to a global maximum point for Ψ . \square

Consider the open subset of $H^s(\mathbb{R}^n)$

$$\mathcal{O} = \left\{ u \in H^s(\mathbb{R}^n) \setminus \{0\} : \int G_\infty(u) > 0 \right\}.$$

Then we have the following

Lemma 4.2. *The map defined by $\mathcal{O} \ni u \mapsto \vartheta_2(u) \in (0, \infty)$, such that $u(\cdot/\vartheta_2(u)) \in \mathcal{P}$, is continuous.*

Proof. Let $u \in \mathcal{O}$ and consider $(u_j) \subset \mathcal{O}$ such that $u_j \rightarrow u$ in $H^s(\mathbb{R}^n)$ as $j \rightarrow \infty$. First note that $\vartheta_2(u_j)$ is bounded. Indeed, consider the expression (4.3) of $\psi' = 0$ in the proof of Lemma 4.1 for u_j and $\vartheta_2(u_j)$

$$\frac{n-2s}{2} \int |(-\Delta)^{s/2} u_j|^2 = n\vartheta_2^{2s}(u_j) \int \left(\left(a(\vartheta_2(u_j)x) + \frac{\nabla a(\vartheta_2(u_j)x) \cdot (\vartheta_2(u_j)x)}{n} \right) F(u_j) - \frac{\lambda u_j^2}{2} \right).$$

Suppose by contradiction that $\vartheta_2(u_j) \rightarrow \infty$ as $j \rightarrow \infty$, along a suitable subsequence. Then, in light of the assumptions on a and F and by Lebesgue Dominated Convergence Theorem, the right-hand side of the above equation goes to infinity while the left-hand converges to $(n-2s)/2 \|(-\Delta)^{s/2} u\|_2^2$, which is a contradiction. Hence, $\vartheta_2(u_j)$ admits a convergent subsequence, say $\vartheta_2(u_j) \rightarrow \bar{\vartheta}$ as $j \rightarrow \infty$. In turn, by Lebesgue Dominated Convergence Theorem, as $j \rightarrow \infty$, we have

$$\begin{aligned} \int a(\vartheta_2(u_j)x) F(u_j) &\rightarrow \int a(\bar{\vartheta}x) F(u), \\ \int \nabla a(\vartheta_2(u_j)x) \cdot (\vartheta_2(u_j)x) F(u_j) &\rightarrow \int \nabla a(\bar{\vartheta}x) \cdot (\bar{\vartheta}x) F(u). \end{aligned}$$

Then, since $u_j \rightarrow u$ in $H^s(\mathbb{R}^n)$ as $j \rightarrow \infty$, we obtain

$$\frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n\bar{\vartheta}^{2s} \int \left(\left(a(\bar{\vartheta}x) + \frac{\nabla a(\bar{\vartheta}x) \cdot (\bar{\vartheta}x)}{n} \right) F(u) - \frac{\lambda u^2}{2} \right).$$

Hence $u(\cdot/\bar{\vartheta}) \in \mathcal{P}$ and, by uniqueness of the projection in \mathcal{P} , $\bar{\vartheta} = \vartheta_2(u)$. \square

Lemma 4.3. *If $u \in \mathcal{P}_\infty$, then $\int G_\infty(u) > 0$.*

Proof. Let $u \in \mathcal{P}_\infty$. Of course $\int G_\infty(u) \geq 0$. Assume by contradiction that $\int G_\infty(u) = 0$. Then

$$0 = \|(-\Delta)^{s/2} u\|_2^2 = \frac{C(n,s)}{2} \iint \frac{(u(x) - u(y))^2}{|x - y|^{n+2s}} dx dy,$$

so that u is constant and hence, as $u \in L^2(\mathbb{R}^n)$, $u = 0$, contradicting $u \in \mathcal{P}_\infty$. \square

Lemma 4.4. *If $u \in \mathcal{P}_\infty$, then there exists a unique $\vartheta > 0$ such that $u(\cdot/\vartheta) \in \mathcal{P}$ and $\vartheta > 1$.*

Proof. Let $u \in \mathcal{P}_\infty$. Then, by Lemma 4.3, $\int G_\infty(u) > 0$. In turn, by Lemma 4.1, there exists a unique $\vartheta > 0$ such that $u(\cdot/\vartheta) \in \mathcal{P}$. Now, we are left with the proof that $\vartheta > 1$. By the arguments in the previous lemmas, it follows that ϑ satisfies

$$\frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n\vartheta^{2s} \int \left(\left(a(\vartheta x) + \frac{\nabla a(\vartheta x) \cdot (\vartheta x)}{n} \right) F(u) - \lambda \frac{u^2}{2} \right).$$

By condition (A4), we get

$$\frac{n-2s}{2n} \int |(-\Delta)^{s/2} u|^2 < \vartheta^{2s} \int \left(a_\infty F(u) - \lambda \frac{u^2}{2} \right) = \vartheta^{2s} \int G_\infty(u).$$

Since $u \in \mathcal{P}_\infty$, the inequality above yields $\vartheta > 1$. \square

Lemma 4.5. *If $u \in \mathcal{P}$, then $\int G_\infty(u) > 0$.*

Proof. Let $u \in \mathcal{P}$. Then, by condition (A4), u satisfies

$$\frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) - \lambda \frac{u^2}{2} \right) < n \int G_\infty(u).$$

Since $\int |(-\Delta)^{s/2} u|^2 > 0$ otherwise u would be constant and hence the zero function as $u \in L^2(\mathbb{R}^n)$, the assertion follows. \square

Lemma 4.6. *If $u \in \mathcal{P}$, then there exists a unique $\vartheta > 0$ such that $u(\cdot/\vartheta) \in \mathcal{P}_\infty$ and $\vartheta < 1$.*

Proof. Let $u \in \mathcal{P}$, then $\int G_\infty(u) > 0$ by Lemma 4.5. By Lemma 4.1, there exists a unique $\vartheta > 0$ such that $u(\cdot/\vartheta) \in \mathcal{P}_\infty$. We are left with the proof that $\vartheta < 1$. Notice that

$$\frac{n-2s}{2n} \int |(-\Delta)^{s/2} u|^2 < \int G_\infty(u).$$

Since $u(\cdot/\vartheta) \in \mathcal{P}_\infty$, then the assertion follows since $\vartheta > 0$ satisfies

$$\vartheta^{2s} = \frac{n-2s}{2n} \frac{\int |(-\Delta)^{s/2} u|^2}{\int G_\infty(u)} < 1.$$

This concludes the proof. \square

Notice that, as a consequence of the previous results, a given function $u \in H^s(\mathbb{R}^n) \setminus \{0\}$ can be projected onto the manifolds \mathcal{P} and \mathcal{P}_∞ if and only if $\int G_\infty(u) > 0$. We will also need the following

Lemma 4.7. *If $u \in \mathcal{P}_\infty$, then $u(\cdot - y) \in \mathcal{P}_\infty$, for all $y \in \mathbb{R}^n$. Moreover, there exists $\vartheta_y > 1$ with*

$$u\left(\frac{\cdot - y}{\vartheta_y}\right) \in \mathcal{P}, \quad \lim_{|y| \rightarrow \infty} \vartheta_y = 1.$$

Proof. If $u \in \mathcal{P}_\infty$, then from translation invariance, we have $u(\cdot - y) \in \mathcal{P}_\infty$, for all $y \in \mathbb{R}^n$. Furthermore, from Lemma 4.4, there exists $\vartheta_y > 1$ such that $u((\cdot - y)/\vartheta_y) \in \mathcal{P}$. Suppose by contradiction that there exists a sequence $(y_j) \subset \mathbb{R}^n$ with $|y_j| \rightarrow +\infty$ and ϑ_{y_j} converges either to $A > 1$ or $+\infty$. Let us define

$$K(\vartheta_{y_j} x + y_j) := a(\vartheta_{y_j} x + y_j) + \frac{\nabla a(\vartheta_{y_j} x + y_j) \cdot (\vartheta_{y_j} x + y_j)}{n}.$$

From conditions (f1)-(f2) we have $0 \leq K(\vartheta_{y_j} x + y_j) F(u(x)) < a_\infty F(u(x)) \leq C u^2(x)$ for a.e. $x \in \mathbb{R}^n$ and for some positive constant C . Hence, by Lebesgue Dominated Convergence Theorem, we get

$$(4.4) \quad \lim_{j \rightarrow \infty} \int \left(K(\vartheta_{y_j} x + y_j) F(u) - \lambda \frac{u^2}{2} \right) = \int G_\infty(u).$$

But for each y_j it follows that $u(\frac{\cdot - y_j}{\vartheta_{y_j}}) \in \mathcal{P}$ with $\vartheta_{y_j} > 1$, which means we have

$$(4.5) \quad \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n \vartheta_{y_j}^{2s} \int \left(K(\vartheta_{y_j} x + y_j) F(u) - \lambda \frac{u^2}{2} \right).$$

The right-hand side of formula (4.5) goes to $+\infty$ or to $n A^{2s} \int G_\infty(u)$, while the left-hand side is constant. In the first case we immediately get a contradiction. In the second case, as $u \in \mathcal{P}_\infty$ and $A > 1$, we get a contradiction too. \square

Under the assumption of Lemma 4.7, we have the following

Lemma 4.8. $\sup_{y \in \mathbb{R}^n} \vartheta_y = \bar{\vartheta} < +\infty$ and $\bar{\vartheta} > 1$.

Proof. From Lemma 4.7 there is $R > 0$ such that $|\vartheta_y| \leq 2$ if $|y| > R$. There exists $M > 0$ such that $\sup\{\vartheta_y : |y| \leq R\} \leq M$. In fact, suppose that there exists a sequence (y_j) with $|y_j| \leq R$ such that $\vartheta_{y_j} \rightarrow +\infty$ as $j \rightarrow \infty$. As in the previous lemma, (4.4) holds. Therefore, from (4.5), it follows

$$\frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 = n\vartheta_{y_j}^{2s} \left(\int G_\infty(u) + o_{y_j}(1) \right).$$

Since $\vartheta_{y_j} \rightarrow +\infty$ and the left-hand side is constant we get a contradiction and the proof is complete. \square

Lemma 4.9. *There exists a real number $\hat{\sigma} > 0$ such that $\inf_{u \in \mathcal{P}} \int |(-\Delta)^{s/2} u|^2 \geq \hat{\sigma}$.*

Proof. Let $u \in \mathcal{P}$, then u satisfies (3.1) and by condition (A4), we have

$$0 < \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 < n \int \left(a_\infty F(u) - \lambda \frac{u^2}{2} \right).$$

On the other hand, from condition (1.8) with $p = 2n/(n-2s)$, given $0 < \varepsilon < \frac{\lambda}{a_\infty}$, we get

$$0 < \frac{n-2s}{2n} \int |(-\Delta)^{s/2} u|^2 < a_\infty C \|u\|_{2n/(n-2s)}^{2n/(n-2s)}.$$

for some $C > 0$. Using the fractional Sobolev inequality (cf. [10, Theorem 6.5]), we find $\hat{C} > 0$ with

$$0 < \frac{n-2s}{2na_\infty C \hat{C}} < \left(\int |(-\Delta)^{s/2} u|^2 \right)^{2s/(n-2s)},$$

which yields the assertion with $\hat{\sigma} := ((n-2s)/(2na_\infty C \hat{C}))^{(n-2s)/2s} > 0$. \square

Lemma 4.10. $p =: \inf_{u \in \mathcal{P}} I(u) > 0$.

Proof. Let $u \in \mathcal{P}$, then $I(u)$ satisfies

$$(4.6) \quad I(u) = \frac{s}{n} \int |(-\Delta)^{s/2} u|^2 + \int \frac{\nabla a(x) \cdot x}{n} F(u) \geq \frac{s}{n} \int |(-\Delta)^{s/2} u|^2 \geq \frac{s\hat{\sigma}}{n} > 0,$$

by Lemma 4.9 and condition (A3). This concludes the proof. \square

If $u \in H^s(\mathbb{R}^n)$ with $\int G_\infty(u) > 0$ and $\vartheta > 0$ is such that $u(\cdot/\vartheta) \in \mathcal{P}_\infty$, then

$$(4.7) \quad I_\infty(u(x/\vartheta)) = \frac{s}{n} \vartheta^{n-2s} \int |(-\Delta)^{s/2} u|^2.$$

Let c_∞ be defined as in (2.3). Then, we have the following

Lemma 4.11. $p = c_\infty$.

Proof. Let $w \in H^s(\mathbb{R}^n)$ be a ground state solution to (2.1). Then $w \in \mathcal{P}_\infty$ and $I_\infty(w) = c_\infty$, by virtue of Theorem 2.1. Set $w_y := w(x-y)$, for any $y \in \mathbb{R}^n$. Of course $w_y \in \mathcal{P}_\infty$ and $I_\infty(w_y) = c_\infty$, by translation invariance. From Lemma 4.4 we find a unique $\vartheta_y > 1$ with $\tilde{w}_y = w_y(\cdot/\vartheta_y) \in \mathcal{P}$.

Therefore, we have

$$\begin{aligned}
|I(\tilde{w}_y) - c_\infty| &= |I(\tilde{w}_y) - I_\infty(w_y)| \\
&= \left| \frac{1}{2} \int |(-\Delta)^{s/2} \tilde{w}_y|^2 - \int G(x, \tilde{w}_y) - \frac{1}{2} \int |(-\Delta)^{s/2} w_y|^2 + \int G_\infty(w_y) \right| \\
&= \left| \frac{1}{2} (\vartheta_y^{n-2s} - 1) \int |(-\Delta)^{s/2} w_y|^2 - \int \left(a(x) F(\tilde{w}_y) - \frac{\lambda \tilde{w}_y^2}{2} \right) + \int \left(a_\infty F(w_y) - \frac{\lambda w_y^2}{2} \right) \right| \\
&= \left| \frac{1}{2} (\vartheta_y^{n-2s} - 1) \int |(-\Delta)^{s/2} w|^2 - \vartheta_y^n \int \left(a(x \vartheta_y + y) F(w) - \frac{\lambda w^2}{2} \right) + \int \left(a_\infty F(w) - \frac{\lambda w^2}{2} \right) \right| \\
&= \left| \frac{1}{2} (\vartheta_y^{n-2s} - 1) \int |(-\Delta)^{s/2} w|^2 + (\vartheta_y^n - 1) \int \frac{\lambda w^2}{2} - \vartheta_y^n \int a(x \vartheta_y + y) F(w) + \int a_\infty F(w) \right| \\
&\leq \frac{|\vartheta_y^{n-2s} - 1|}{2} \int |(-\Delta)^{s/2} w|^2 + |\vartheta_y^n - 1| \int \frac{\lambda w^2}{2} + \int |F(w)| |a_\infty - \vartheta_y^n a(x \vartheta_y + y)|.
\end{aligned}$$

Since $\vartheta_y \rightarrow 1$ if $|y| \rightarrow +\infty$, we obtain

$$|F(w)| |a_\infty - \vartheta_y^n a(x \vartheta_y + y)| \rightarrow 0 \quad \text{as } |y| \rightarrow \infty, \text{ a.e. in } \mathbb{R}^n, \quad |F(w)| |a_\infty - \vartheta_y^n a(x \vartheta_y + y)| \leq C|w|^2,$$

for some positive constant C independent of y . By Lebesgue Dominated Convergence Theorem,

$$\int |F(w)| |a_\infty - \vartheta_y^n a(x \vartheta_y + y)| = o_y(1), \quad \text{as } |y| \rightarrow \infty.$$

In turn, we conclude that $|I(\tilde{w}_y) - c_\infty| \leq o_y(1)$, as $|y| \rightarrow \infty$. Then, $p = \inf_{u \in \mathcal{P}} I(u) \leq c_\infty$. On the other hand, consider $u \in \mathcal{P}$ and let $0 < \vartheta < 1$ by Lemma 4.6 be such that $u(\cdot/\vartheta) \in \mathcal{P}_\infty$. Since $u \in \mathcal{P}$, then

$$\begin{aligned}
I(u) &= \frac{s}{n} \int |(-\Delta)^{s/2} u|^2 + \frac{1}{n} \int \nabla a(x) \cdot x F(u) > \frac{s}{n} \int |(-\Delta)^{s/2} u|^2 \\
&\geq \frac{s}{n} \vartheta^{n-2s} \int |(-\Delta)^{s/2} u|^2 = I_\infty(u(x/\vartheta)) \geq \inf_{u \in \mathcal{P}_\infty} I_\infty(u) = m = c_\infty,
\end{aligned}$$

in light of (4.7), (A3) and Lemma 2.4. Hence $p = \inf_{u \in \mathcal{P}} I(u) \geq c_\infty$, which concludes the proof. \square

Lemma 4.12. \mathcal{P} is a natural constraint for (1.2).

Proof. If $u \in \mathcal{P}$ is a critical point of $I|_{\mathcal{P}}$, there exists $\mu \in \mathbb{R}$ with $I'(u) + \mu J'(u) = 0$. The proof is complete as soon as we show that $\mu = 0$. Computing $I'(u)(\varphi) + \mu J'(u)(\varphi)$ for any $\varphi \in H^s(\mathbb{R}^n)$ yields

$$\begin{aligned}
0 &= \int (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi + \lambda u \varphi - \int a(x) f(u) \varphi \\
&\quad + \mu \left[(n-2s) \int (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi - n \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) f(u) \varphi - \lambda u \varphi \right) \right].
\end{aligned}$$

so that u satisfies the equation

$$(1 + \mu(n-2s))(-\Delta)^s u + \lambda(1 + \mu n)u = [(1 + \mu n)a(x) + \mu \nabla a(x) \cdot x] f(u).$$

The solutions of this equation satisfy a Pohožaev identity $Q(u) = 0$, where

$$Q(u) = \frac{(1 + \mu(n-2s))(n-2s)}{2} \int |(-\Delta)^{s/2} u|^2 - n \int \widehat{G}(x, u) - \int x \cdot \widehat{G}_x(x, u),$$

where we have

$$\begin{aligned}\widehat{G}(x, u) &= ((1 + \mu n)a(x) + \mu \nabla a(x) \cdot x) F(u) - \lambda \frac{(1 + \mu n)}{2} u^2, \\ x \cdot \widehat{G}_x(x, u) &= ((1 + \mu + \mu n) \nabla a(x) \cdot x + \mu x \cdot \mathcal{H}_a(x) \cdot x) F(u).\end{aligned}$$

Therefore, Q rewrites as follows

$$\begin{aligned}Q(u) &= \frac{(1 + \mu(n - 2s))(n - 2s)}{2} \int |(-\Delta)^{s/2} u|^2 \\ &\quad - n \int ((1 + \mu n)a(x) + \mu \nabla a(x) \cdot x) F(u) - \lambda \frac{(1 + \mu n)}{2} u^2 \\ &\quad - \int ((1 + \mu + \mu n) \nabla a(x) \cdot x + \mu x \cdot \mathcal{H}_a(x) \cdot x) F(u) \\ &= \frac{(1 + \mu(n - 2s))(n - 2s)}{2} \int |(-\Delta)^{s/2} u|^2 \\ &\quad - n(1 + \mu n) \int \left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(u) - \lambda \frac{u^2}{2} \\ &\quad - (n + 1)\mu \int \left(\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n + 1} \right) F(u).\end{aligned}$$

Recalling that $u \in \mathcal{P}$ and substituting (3.1) in the equation above, it follows that

$$\begin{aligned}Q(u) &= \frac{(1 + \mu(n - 2s))(n - 2s)}{2} \int |(-\Delta)^{s/2} u|^2 - (1 + \mu n) \frac{(n - 2s)}{2} \int |(-\Delta)^{s/2} u|^2 \\ &\quad - (n + 1)\mu \int \left(\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n + 1} \right) F(u) \\ &= -\mu s(n - 2s) \int |(-\Delta)^{s/2} u|^2 - (n + 1)\mu \int \left(\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n + 1} \right) F(u).\end{aligned}$$

On the other hand, since u satisfies $Q(u) = 0$, we end up with

$$-\mu s(n - 2s) \int |(-\Delta)^{s/2} u|^2 = (n + 1)\mu \int \left(\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n + 1} \right) F(u).$$

From (A5) we have that, if $\mu > 0$, the right-hand side of the equation is nonnegative as

$$\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n + 1} \geq \frac{n}{n + 1} \left(\nabla a(x) \cdot x + \frac{x \cdot \mathcal{H}_a(x) \cdot x}{n} \right) \geq 0,$$

while the left-hand side is negative. If $\mu < 0$ one gets the same contradiction. Whence $\mu = 0$. \square

Proof of Theorem 1.1 concluded. Assume by contradiction that there exists a critical point $z \in H^s(\mathbb{R}^n)$ of I at level p . In particular, $z \in \mathcal{P}$ and $I(z) = p$. Let $\vartheta \in (0, 1)$ be such that $z(\cdot/\vartheta) \in \mathcal{P}_\infty$. Then

$$\begin{aligned}p = I(z) &= \frac{s}{n} \int |(-\Delta)^{s/2} z|^2 + \frac{1}{n} \int \nabla a(x) \cdot x F(z) \\ &> \frac{s}{n} \int |(-\Delta)^{s/2} z|^2 > \frac{s}{n} \vartheta^{n-2s} \int |(-\Delta)^{s/2} z|^2 \\ &= I_\infty(z(\cdot/\vartheta)) \geq \inf_{u \in \mathcal{P}_\infty} I_\infty(u) = m = c_\infty,\end{aligned}$$

using (A3) and (4.7), Lemma 2.4 and Theorem 2.1. Then $p > c_\infty$, contradicting Lemma 4.11. In particular the infimum p is not achieved, otherwise, if $I(v) = p$ and $I'|_{\mathcal{P}}(v) = 0$ for some $v \in H^s(\mathbb{R}^n)$, in light of Lemma 4.12, we would have $I'(v) = 0$, contradicting the first part of Theorem 1.1. \square

5. EXISTENCE RESULTS

In this section we show the existence of a solution of problem (1.2). To this aim, we shall assume that the hypotheses of Theorem 1.2 are satisfied. As we have seen in the previous sections, we should look for solutions which have energy levels above c_∞ . In order to find such a solution we follow some ideas of [1] based upon linking and the barycenter function on the Nehari manifold. In our case, since the nonlinear terms of the equation are not homogeneous, we are led to the Pohožaev manifold \mathcal{P} and obtain the desired solution by a linking argument. We also make use of a barycenter function, similar to that of [1] and used by G.S. Spradlin [25, 26] as well.

Lemma 5.1. *I satisfies the geometrical properties of the Mountain Pass theorem.*

Proof. On one hand, for the local minimum condition at the origin, by (1.8) one can argue exactly as in the proof of Lemma 2.3. On the other hand, if $w \in H^s(\mathbb{R}^n)$ is a least energy solution to (2.1), by Lemma 2.2 there exists $\gamma \in \Gamma_\infty$ such that $\gamma(t) = w(x/tL)$ for $t > 0$ and $L > 0$ large enough. In turn, if $\gamma_y(t) := w((\cdot - y)/tL)$, by (A2) and Lebesgue Dominated Convergence Theorem,

$$I(\gamma_y(1)) = I_\infty(\gamma_y(1)) + \int (a_\infty - a(x+y))F(\gamma(1)) = I_\infty(\gamma(1)) + o_y(1) < 0, \quad \text{for } |y| \text{ large,}$$

since $I_\infty(\gamma(1)) < 0$, concluding the proof. \square

Let c be the min-max mountain pass level for I

$$(5.1) \quad c = \min_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1], H^s(\mathbb{R}^n)) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

We start by proving that the min-max levels of the Mountain Pass Theorem for I and I_∞ agree.

Lemma 5.2. $c_\infty = c$.

Proof. If $\gamma \in \Gamma$, then $I(\gamma(1)) < 0$ and since $I_\infty \leq I$, we have $I_\infty(\gamma(1)) < 0$. Then, $\Gamma \subset \Gamma_\infty$ yielding

$$c_\infty = \inf_{\gamma \in \Gamma_\infty} \max_{t \in [0,1]} I_\infty(\gamma(t)) \leq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\infty(\gamma(t)) \leq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) = c.$$

Let now $\varepsilon > 0$ be arbitrary and let $\gamma \in \Gamma_\infty$ such that $I_\infty(\gamma(t)) \leq c_\infty + \varepsilon$, for all $t \in [0,1]$. Choose $y \in \mathbb{R}^n$ and translating $\tau_y(\gamma(t))(x) := \gamma(t)(x - y)$ with $|y|$ large enough, we get $\tau_y \circ \gamma \in \Gamma$ (see Lemma 5.1). If $t_y \in [0,1]$ is such that $I(\tau_y(\gamma(t_y)))$ is the maximum value on $[0,1]$ of $t \mapsto I(\tau_y \circ \gamma(t))$, then

$$c_\infty + \varepsilon \geq I_\infty(\gamma(t_y)) = I_\infty(\tau_y \circ \gamma(t_y)) = \max_{[0,1]} I(\tau_y \circ \gamma) \geq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) = c.$$

This gives $c_\infty \geq c$ by the arbitrariness of ε and the assertion follows. \square

Lemma 5.3. $p = c$.

Proof. The assertion follows by combining Lemma 4.11 and Lemma 5.2. \square

Now we observe the following property of \mathcal{P} with respect to the paths in the Mountain Pass Theorem.

Lemma 5.4. *For every $\gamma \in \Gamma$ there exists $s \in (0,1)$ such that $\gamma(s) \in \mathcal{P}$.*

Proof. By the proof of Lemma 3.1 (a), we learn that there exists $\rho > 0$ such that $J(u) > 0$ if $0 < \|u\|_{H^s} < \rho$. Furthermore, we have

$$\begin{aligned} J(u) &= \frac{n-2s}{2} \int |(-\Delta)^{s/2} u|^2 - n \int G(x, u) - \int \nabla a(x) \cdot x F(u) \\ &= nI(u) - s \int |(-\Delta)^{s/2} u|^2 - \int \nabla a(x) \cdot x F(u). \end{aligned}$$

From (A3) it follows that $J(u) < nI(u)$, for every $u \in H^s(\mathbb{R}^n) \setminus \{0\}$. If $\gamma \in \Gamma$, we have $J(\gamma(0)) = 0$ and $J(\gamma(1)) < nI(\gamma(1)) < 0$. Then there exists $t \in (0,1)$ with $\|\gamma(t)\|_{H^s} > \rho$ and $J(\gamma(t)) = 0$. \square

We recall that a sequence (u_j) is said to be a Cerami sequence for I at level d in \mathbb{R} , denoted by $(Ce)_d$, if $I(u_n) \rightarrow d$ and $\|I'(u_j)\|_{H^{-s}}(1 + \|u_j\|_{H^s}) \rightarrow 0$. We have the following

Lemma 5.5. *If (u_j) is a $(Ce)_d$ sequence with $d > 0$, then it has a bounded subsequence.*

Proof. By contradiction, let $\|u_j\|_{H^s} \rightarrow +\infty$. If $\hat{u}_j := u_j \|u_j\|_{H^s}^{-1}$, then $\|\hat{u}_j\|_{H^s} = 1$ and $\hat{u}_j \rightharpoonup \hat{u}$, up to a subsequence. Therefore, one of the two cases occur:

$$\text{Case 1 : } \limsup_{j \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B_1(y)} |\hat{u}_j|^2 = \delta > 0,$$

$$\text{Case 2 : } \limsup_{j \rightarrow \infty} \sup_{y \in \mathbb{R}^n} \int_{B_1(y)} |\hat{u}_j|^2 = 0.$$

Suppose Case 2 hold. Fixing $L > 2\sqrt{dD}$, with D as in assumption (f3), gives

$$I(Lu_j \|u_j\|_{H^s}^{-1}) = \frac{L^2}{2} - \int a(x) F(Lu_j \|u_j\|_{H^s}^{-1}).$$

Given $\varepsilon > 0$, by inequality (1.8) there exists $C_\varepsilon > 0$ (here $2 < p < 2n/(n - 2s)$) with

$$\int a(x) F(Lu_j \|u_j\|_{H^s}^{-1}) < \frac{a_\infty \varepsilon L^2}{2} \frac{\|u_j\|_2^2}{\lambda \|u_j\|_2^2 + \|(-\Delta)^{s/2} u_j\|_2^2} + C_\varepsilon L^p \|\hat{u}_j\|_p^p \leq \frac{a_\infty \varepsilon L^2}{2\lambda} + o_j(1),$$

where $\|\hat{u}_j\|_p \rightarrow 0$ by a variant of Lions' Lemma [19, Lemma I.1]. For $\varepsilon = \lambda/(2a_\infty)$, we have

$$I(Lu_j \|u_j\|_{H^s}^{-1}) \geq \frac{L^2}{4} - o_j(1).$$

We have $L\|u_j\|_{H^s}^{-1} \in (0, 1)$ for j large and if we consider $t_j \in (0, 1)$ with $I(t_j u_j) = \max_{t \in [0, 1]} I(tu_j)$,

$$(5.2) \quad I(t_j u_j) = \max_{t \in [0, 1]} I(tu_j) \geq I(Lu_j \|u_j\|_{H^s}^{-1}) \geq \frac{L^2}{4} - o_j(1).$$

On the other hand, using (f3) we obtain

$$(5.3) \quad \begin{aligned} I(t_j u_j) &= I(t_j u_j) - \frac{1}{2} I'(t_j u_j)(t_j u_j) = \int a(x) \left(\frac{1}{2} f(t_j u_j)(t_j u_j) - F(t_j u_j) \right) \\ &\leq D \int a(x) \left(\frac{1}{2} f(u_j) u_j - F(u_j) \right) = D(I(u_j) - \frac{1}{2} I'(u_j) u_j) = Dd + o_j(1). \end{aligned}$$

Then, on account of the choice of L , combining (5.2) and (5.3), we get a contradiction. In Case 1, let (y_j) be a sequence such that $|y_j| \rightarrow +\infty$ and

$$(5.4) \quad \int_{B_1(y_j)} |\hat{u}_j|^2 > \delta/2.$$

Recalling that $\hat{u}_j(\cdot + y_j) \rightharpoonup \bar{u}$ in $H^s(\mathbb{R}^n)$ as $j \rightarrow \infty$, we obtain $\int_{B_1(0)} |\bar{u}(x)|^2 > \delta/2$, namely $\bar{u} \neq 0$. Thus, there exists $\Omega \subset B_1(0)$, with $|\Omega| > 0$ such that

$$(5.5) \quad 0 \neq \bar{u}(x) = \lim_{j \rightarrow \infty} \hat{u}_j(x + y_j) = \lim_{j \rightarrow \infty} \frac{u_j(x + y_j)}{\|u_j\|_{H^s}}, \quad \text{a.e. } x \in \Omega,$$

yielding $u_j(x + y_j) \rightarrow \infty$ for a.e. $x \in \Omega$. We claim that, actually $u_j(x + y_j) \rightarrow +\infty$ for $x \in \Omega$. Setting $\zeta_j(x) := \hat{u}_j(x + y_j)$, for a $\mu_j \rightarrow 0$ in $H^{-s}(\mathbb{R}^n)$ as $j \rightarrow \infty$, we have

$$(-\Delta)^{s/2} \zeta_j + \lambda \zeta_j = \frac{a(x + y_j)}{\|u_j\|_{H^s}} f(\|u_j\|_{H^s} \zeta_j) + \frac{\mu_j}{\|u_j\|_{H^s}}.$$

Testing this equation by ζ_j^- and taking into account that

$$\int \frac{a(x+y_j)}{\|u_j\|_{H^s}} f(\|u_j\|_{H^s} \zeta_j) \zeta_j^- = 0, \quad \frac{\langle \mu_j, \zeta_j^- \rangle}{\|u_j\|_{H^s}} = \frac{\langle \mu_j, u_j^-(\cdot + y_j) \rangle}{\|u_j\|_{H^s}^2} = o_j(1),$$

by arguing as around formula (1.10), we conclude that $\|\zeta_j^-\|_{H^s} = o_j(1)$ as $j \rightarrow \infty$, hence in particular by the fractional Sobolev embedding $\|\zeta_j^-\|_{L^p} = o_j(1)$ as $j \rightarrow \infty$ for any $2 \leq p \leq 2n/(n-2s)$. Since $\zeta_j = \hat{u}_j(\cdot + y_j) \rightarrow \bar{u}$ in $L^p(\Omega)$, we also have $\zeta_j^- = \hat{u}_j^-(\cdot + y_j) \rightarrow \bar{u}^-$ in $L^p(\Omega)$. But then $\bar{u}^- = 0$ on Ω which means $\bar{u} > 0$ on Ω . In turn, from 5.5, we have the claim. Thus, by (f3), Fatou Lemma and (A1), with $\sigma := \inf_{\mathbb{R}^n} a$,

$$\begin{aligned} & \liminf_{j \rightarrow \infty} \int a(x) \left(\frac{1}{2} f(u_j) u_j - F(u_j) \right) \\ &= \liminf_{j \rightarrow \infty} \int a(x+y_j) \left(\frac{1}{2} f(u_j(x+y_j)) u_j(x+y_j) - F(u_j(x+y_j)) \right) \\ &\geq \liminf_{j \rightarrow \infty} \int_{\Omega} \sigma \left(\frac{1}{2} f(u_j(x+y_j)) u_j(x+y_j) - F(u_j(x+y_j)) \right) \\ &\geq \int_{\Omega} \liminf_{j \rightarrow \infty} \sigma \left(\frac{1}{2} f(u_j(x+y_j)) u_j(x+y_j) - F(u_j(x+y_j)) \right) = +\infty. \end{aligned}$$

On the other hand, $|I'(u_j)u_j| \leq \|I'(u_j)\|_{H^{-s}} \|u_j\|_{H^s} \rightarrow 0$, as $j \rightarrow \infty$. Then,

$$\int a(x) \left(\frac{1}{2} f(u_j) u_j - F(u_j) \right) = I(u_j) - \frac{1}{2} I'(u_j)u_j = d + o_j(1),$$

which gives a contradiction. If, instead, (y_j) in (5.4) is bounded, say $|y_j| \leq R$ for some R , we obtain

$$\frac{\delta}{2} \leq \int_{B_1(y_j)} |\hat{u}_j|^2 \leq \int_{B_{2R}(0)} |\hat{u}_j|^2,$$

and since $\hat{u}_j \rightarrow \hat{u}$ in $L^2(B_{2R}(0))$, it follows that

$$\delta/2 \leq \int_{B_{2R}(0)} |\hat{u}|^2.$$

Similarly to the previous case, there exists $\Omega \subset B_{2R}(0)$ of positive measure such that (5.5) holds. The argument follows as above for the case where (y_j) is unbounded and we get a contradiction. \square

The next step is to show the existence of a Cerami sequence for the functional I at level c .

Lemma 5.6. *Let c be as in (5.1), then there exists a $(Ce)_c$ sequence $(u_n) \subset H^s(\mathbb{R}^n)$.*

Proof. We apply the Ghoussoub-Preiss theorem [11, Theorem 6] with $X = H^s(\mathbb{R}^n)$, see also [15]. Consider $z_0 = 0$ and z_1 in $H^s(\mathbb{R}^n)$ with $I(z_1) < 0$ (cf. Lemma 5.1). Then the Pohožaev manifold \mathcal{P} separates z_0 and z_1 . Indeed, observe that $z_0 = 0 \notin \mathcal{P}$ and $z_1 \notin \mathcal{P}$, since $J(z_1) < nI(z_1) < 0$ (cf. proof of (a) of Lemma 5.4). Moreover, there exists $\rho > 0$ such that, if $0 < \|u\|_{H^s} < \rho$, then $J(u) > 0$ (cf. proof of Lemma 3.1). We have $H^s(\mathbb{R}^n) \setminus \mathcal{P} = \{0\} \cup \{J > 0\} \cup \{J < 0\}$. The ball $B_\rho(z_0)$ is in a connected component C_1 of $\{0\} \cup \{J > 0\}$. On the other hand, z_1 is in a connected component of $\{J < 0\}$. In this setting, we get a sequence $(u_j) \subset H^s(\mathbb{R}^n)$ such that

$$\delta(u_j, \mathcal{P}) \rightarrow 0, \quad I(u_j) \rightarrow c, \quad \|I'(u_j)\|(1 + \|u_j\|_{H^s}) \rightarrow 0,$$

where δ denotes the geodesic metric on $H^s(\mathbb{R}^n)$, defined by

$$\delta(u, v) := \inf \left\{ \int_0^1 \frac{\|\gamma'(\sigma)\|_{H^s}}{1 + \|\gamma(\sigma)\|_{H^s}} d\sigma : \gamma \in C^1([0, 1], H^s(\mathbb{R}^n)), \gamma(0) = u, \gamma(1) = v \right\}.$$

This completes the proof. \square

For the following type of properties, we refer the reader to the book [29].

Lemma 5.7. *Let $(u_j) \in H^s(\mathbb{R}^n)$ be a bounded sequence such that*

$$I(u_j) \rightarrow d > 0 \quad \text{and} \quad \|I'(u_j)\|_{H^{-s}}(1 + \|u_j\|_{H^s}) \rightarrow 0.$$

Replacing (u_j) by a subsequence, if necessary, there exists a solution \bar{u} of (1.2), a number $k \in \mathbb{N} \cup \{0\}$, k functions u^1, u^2, \dots, u^k and k sequences of points $y_j^1, y_j^2, \dots, y_j^k \in \mathbb{R}^n$, satisfying:

- a) $u_j \rightarrow \bar{u}$ in $H^s(\mathbb{R}^n)$ or
- b) $u^i \in H^s(\mathbb{R}^n)$ are positive solutions to (2.1) radially symmetric about some point;
- c) $|y_n^i| \rightarrow +\infty$ and $|y_n^i - y_n^m| \rightarrow +\infty$, $i \neq m$;
- d) $u_j - \sum_{i=1}^k u^i(x - y_j^i) \rightarrow \bar{u}$;
- e) $I(u_j) \rightarrow I(\bar{u}) + \sum_{i=1}^k I_\infty(u^i)$.

That the solutions $u^i \in H^s(\mathbb{R}^n)$ to (2.1) are positive and radially symmetric about some point follows from [12, Theorem 1.3], namely a Gidas-Ni-Nirenberg type result in the fractional case ($u^i \neq 0$, $u^i \geq 0$ and hence $u^i > 0$, see [12]).

Corollary 5.8. *If $I(u_j) \rightarrow c_\infty$ and $\|I'(u_j)\|_{H^{-s}}(1 + \|u_j\|_{H^s}) \rightarrow 0$, then either (u_j) is relatively compact in $H^s(\mathbb{R}^n)$ or Lemma 5.7 holds with $k = 1$ and $\bar{u} = 0$.*

Let us set

$$c_\# := \inf \{c > c_\infty : c \text{ is a radial critical value of } I_\infty\}.$$

Then we have the following

Lemma 5.9. *Assume that*

(5.6) c_∞ *is an isolated radial critical level for* I_∞ ,

Then $c_\# > c_\infty$ and I satisfies condition (Ce) at level $d \in (c_\infty, \min\{c_\#, 2c_\infty\})$. Assume now that

(5.7) *the limiting problem (2.1) admits a unique positive radial solution.*

Then I satisfies condition (Ce) at level $d \in (c_\infty, 2c_\infty)$.

Proof. Take a sequence $(u_j) \in H^s(\mathbb{R}^n)$ such that $I(u_j) \rightarrow d$ and $\|I'(u_j)\|_{H^{-s}}(1 + \|u_j\|_{H^s}) \rightarrow 0$ as $j \rightarrow \infty$. By Lemma 5.5, (u_j) has a bounded subsequence. Applying Lemma 5.7, up to subsequences, we have

$$u_j - \sum_{i=1}^k u^i(x - y_j^i) \rightarrow \bar{u} \quad \text{in } H^s(\mathbb{R}^n), \quad I(u_j) \rightarrow I(\bar{u}) + \sum_{i=1}^k I_\infty(u^i),$$

where u^i is a solution to (2.1), $|y_j^i| \rightarrow +\infty$ and \bar{u} is a (possibly zero) solution of (1.2). Since $d < 2c_\infty$, then $k < 2$. If $k = 1$, we have two cases to distinguish.

Let us first assume that (5.6) holds. Then $c_\# > c_\infty$, otherwise there exists a sequence c_j of radially symmetric (about some point) critical values of I_∞ such that $c_j > c_\infty$ and $c_j \rightarrow c_\infty$ as $j \rightarrow \infty$.

- $\bar{u} \neq 0$, which implies $I(\bar{u}) \geq p = c_\infty$ and hence $I(u_j) \geq 2c_\infty$.
- $\bar{u} = 0$, which yields $I(u_j) \rightarrow I_\infty(u_1)$. If $I_\infty(u_1) = c_\infty$, we have a contradiction. If $I_\infty(u_1) = \tilde{c} > c_\infty$, then $I_\infty(u_1) \geq c_\# \geq \min\{c_\#, 2c_\infty\}$, against $d < \min\{c_\#, 2c_\infty\}$. Then $k = 0$ and $u_j \rightarrow \bar{u}$.

Let us now assume that (5.7) holds.

- $\bar{u} \neq 0$, which implies $I(\bar{u}) \geq p = c_\infty$ and hence $I(u_j) \geq 2c_\infty$.
- $\bar{u} = 0$, which yields $I(u_j) \rightarrow I_\infty(u_1) = c_\infty$. The fact that $I_\infty(u_1) = c_\infty$ follows by using uniqueness assumption (5.7). These conclusions go against the assumption $c_\infty < d < 2c_\infty$. \square

Lemma 5.10. *Let $I(u_j) \rightarrow d > 0$ and $\{u_j\} \subset \mathcal{P}$. Then $\{u_j\}$ is bounded in $H^s(\mathbb{R}^n)$.*

Proof. If $u_j \in \mathcal{P}$, then using (A3) and the first equality of (4.6), we get

$$d + 1 \geq I(u_j) \geq \frac{s}{n} \int |(-\Delta)^{s/2} u_j|^2.$$

In turn, by the fractional Sobolev inequality, the sequence $\|u_j\|_{2n/(n-2s)}$ is also bounded. By (1.8) with $\varepsilon < \lambda/\|a\|_\infty$, we have

$$\int a(x)F(u_j) \leq \frac{1}{2}\varepsilon\|a\|_\infty\|u_j\|_2^2 + C_\varepsilon\|u_j\|_{2n/(n-2s)}^{2n/(n-2s)}.$$

Replacing this in the expression of I

$$d + 1 \geq I(u_j) \geq \frac{1}{2} \int |(-\Delta)^{s/2} u_j|^2 + \frac{1}{2}(\lambda - \varepsilon\|a\|_\infty)\|u_j\|_2^2 - C_\varepsilon\|u_j\|_{2n/(n-2s)}^{2n/(n-2s)},$$

so $\|u_j\|_2$ is bounded as well, and the assertion follows. \square

Next, we introduce the barycenter function.

Definition 5.11. Define the barycenter function of a $u \in H^s(\mathbb{R}^n) \setminus \{0\}$ by setting

$$\mu(u)(x) := \frac{1}{|B_1|} \int_{B_1(x)} |u(y)| dy.$$

It follows that $\mu(u) \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Subsequently, take

$$\hat{u}(x) := \left[\mu(u)(x) - \frac{1}{2} \max \mu(u) \right]^+.$$

It follows that $\hat{u} \in C_0(\mathbb{R}^n)$. Now define the barycenter of u by

$$\beta(u) = \frac{1}{\|\hat{u}\|_{L^1}} \int x \hat{u}(x) dx \in \mathbb{R}^n.$$

Since \hat{u} has compact support, by definition, $\beta(u)$ is well defined. β satisfies the following properties:

- (a) β is a continuous function in $H^s(\mathbb{R}^n) \setminus \{0\}$.
- (b) If u is radially symmetric, then $\beta(u) = 0$.
- (c) Given $y \in \mathbb{R}^n$ and setting $u_y(x) := u(x - y)$, then $\beta(u_y) = \beta(u) + y$.

We shall also need the following

Lemma 5.12. *Assume that $u_j, v_j \in H^s(\mathbb{R}^n)$ are such that $\|u_j - v_j\|_{H^s} \rightarrow 0$ and $I'(v_j) \rightarrow 0$ as $j \rightarrow \infty$. Then, $I'(u_j) \rightarrow 0$ as $j \rightarrow \infty$*

Proof. By assumption (1) of Theorem 1.2, we have $f \in \text{Lip}(\mathbb{R}, \mathbb{R}^+)$. Observe first that, for every $w, \varphi, \psi \in H^s(\mathbb{R}^n)$, we have

$$(5.8) \quad I''(w)(\varphi, \psi) = \int (-\Delta)^{s/2} \varphi (-\Delta)^{s/2} \psi + \lambda \int \varphi \psi - \int a(x) f'(w) \varphi \psi.$$

Also, by the Mean Value Theorem, for any $u, v \in H^s(\mathbb{R}^n)$ and $\varphi \in H^s(\mathbb{R}^n)$, there exists $\xi \in (0, 1)$ with

$$I'(v)(\varphi) - I'(u)(\varphi) = I''(u + \xi(v - u))(\varphi, v - u).$$

Therefore, by taking into account that $|f'(u_j + \xi_j(v_j - u_j))| \leq C$ a.e. and for every $j \geq 1$ by assumption (f1), for all $j \geq 1$ we find $\xi_j \in (0, 1)$ such that from formula (5.8) we obtain

$$\begin{aligned} I'(v_j)(\varphi) - I'(u_j)(\varphi) &= I''(u_j + \xi_j(v_j - u_j))(\varphi, v_j - u_j) \\ &= \int (-\Delta)^{s/2} \varphi (-\Delta)^{s/2} (v_j - u_j) + \lambda \int \varphi (v_j - u_j) \\ &\quad - \int a(x) f'(u_j + \xi_j(v_j - u_j)) \varphi (v_j - u_j) \\ &\leq C \|\varphi\|_{H^s} \|v_j - u_j\|_{H^s} + C a_\infty \int |\varphi| |v_j - u_j| \leq C \|\varphi\|_{H^s} \|v_j - u_j\|_{H^s}. \end{aligned}$$

In turn, taking the supremum over the $\varphi \in H^s(\mathbb{R}^n)$ with $\|\varphi\|_{H^s} \leq 1$, we get as $j \rightarrow \infty$

$$\|I'(v_j) - I'(u_j)\|_{H^{-s}} \leq C \|v_j - u_j\|_{H^s} = o_j(1),$$

which concludes the proof. \square

Now we define

$$(5.9) \quad b := \inf \{I(u) : u \in \mathcal{P} \text{ and } \beta(u) = 0\}.$$

It is clear that $b \geq c_\infty$. Moreover, we have the following

Lemma 5.13. $b > c_\infty$.

Proof. Suppose $b = c_\infty$. By definition, there exists a sequence $\{u_j\}$ with $u_j \in \mathcal{P}$ and $\beta(u_j) = 0$ such that $I(u_j) \rightarrow b$. By Lemma 5.10, $\{u_j\}$ is bounded. Since $b = p$ by Lemmas 5.2 and 5.3, then $\{u_j\}$ is also a minimizing sequence of I on \mathcal{P} . By Ekeland Variational Principle, there exists another sequence $\{\tilde{u}_j\} \subset \mathcal{P}$ such that $I(\tilde{u}_j) \rightarrow p$, $I'(\tilde{u}_j) \rightarrow 0$ and $\|\tilde{u}_j - u_j\|_{H^s} \rightarrow 0$ as $j \rightarrow \infty$. Let us now prove that $I'(\tilde{u}_j) \rightarrow 0$, as $j \rightarrow \infty$. Suppose by contradiction that this is not the case. Then, there exists $\sigma > 0$ and a subsequence $\{\tilde{u}_{j_k}\}$ with

$$\|I'(\tilde{u}_{j_k})\| > \sigma, \quad \text{for all } k \geq 1 \text{ large.}$$

Arguing as in the proof of Lemma 5.12, there exists a positive constant C such that

$$|I'(\tilde{u}_{j_k})(\varphi) - I'(v)(\varphi)| \leq C \|\tilde{u}_{j_k} - v\|_{H^s} \|\varphi\|_{H^s}, \quad \text{for all } k \geq 1 \text{ and any } v, \varphi \in H^s(\mathbb{R}^n).$$

Taking the supremum over $\|\varphi\|_{H^s} \leq 1$ yields $\|I'(\tilde{u}_{j_k}) - I'(v)\|_{H^{-s}} \leq C \|\tilde{u}_{j_k} - v\|_{H^s}$ for all $k \geq 1$ and any $v \in H^s(\mathbb{R}^n)$. Therefore, if $\|\tilde{u}_{j_k} - v\|_{H^s} < \tilde{\delta}/C := 2\delta$, then we have $\|I'(\tilde{u}_{j_k}) - I'(v)\|_{H^{-s}} < \tilde{\delta}$, for all $v \in H^s(\mathbb{R}^n)$ and $k \geq 1$. This yields, $\sigma - \tilde{\delta} < \|I'(\tilde{u}_{j_k})\|_{H^{-s}} - \tilde{\delta} < \|I'(v)\|_{H^{-s}}$, for all $k \geq 1$ large. For $\tilde{\delta} \in (0, \sigma)$, we have $\lambda := \sigma - \tilde{\delta} > 0$ and

$$\forall v \in H^s(\mathbb{R}^n) : \quad v \in B_{2\delta}(\tilde{u}_{j_k}) \implies \|I'(v)\|_{H^{-s}} > \lambda.$$

Let us now set $\varepsilon := \min\{p/2, \lambda\delta/8\}$ and $S := \{\tilde{u}_{j_k}\}$. Then, by virtue of [30, Lemma 2.3], there is a deformation $\eta : [0, 1] \times H^s(\mathbb{R}^n) \rightarrow H^s(\mathbb{R}^n)$ at the level p , such that

$$\eta(1, I^{p+\varepsilon} \cap S) \subset I^{p-\varepsilon}, \quad I(\eta(1, u)) \leq I(u), \quad \text{for all } u \in H^s(\mathbb{R}^n).$$

For k large enough, since \tilde{u}_{j_k} is minimizing for p , we have

$$(5.10) \quad \max_{t>0} I(\tilde{u}_{j_k}(\cdot/t)) = I(\tilde{u}_{j_k}) < p + \varepsilon.$$

Observe that, for each $k \geq 1$, by (A4) we have

$$\int G_\infty(\tilde{u}_{j_k}) \geq \int \left(\left(a(x) + \frac{\nabla a(x) \cdot x}{n} \right) F(\tilde{u}_{j_k}) - \lambda \frac{\tilde{u}_{j_k}^2}{2} \right) = \frac{n-2s}{2n} \int |(-\Delta)^{s/2} \tilde{u}_{j_k}|^2 > 0,$$

so that the arguments of Lemma 4.1 work for \tilde{u}_{j_k} . Since $\tilde{u}_{j_k} \in \mathcal{P}$, the first equality in (5.10) is justified by means of formula (4.2) of Lemma 4.1 on Ψ' , by the uniqueness of positive zeros of Ψ' and since $\Psi(\vartheta) > 0$ for ϑ small and $\Psi(\vartheta) < 0$ for ϑ large. Then, we can infer that

$$\max_{t>0} I(\eta(1, \tilde{u}_{j_k}(\cdot/t))) < p - \varepsilon.$$

On the other hand, for k and L fixed large, $\gamma(t) := \eta(1, \tilde{u}_{j_k}(\cdot/Lt))$ is a path in Γ since by (4.1)

$$\begin{aligned} I(\gamma(1)) &= I(\eta(1, \tilde{u}_{j_k}(\cdot/L))) \leq I(\tilde{u}_{j_k}(\cdot/L)) = \frac{L^{n-2s}}{2} \int |(-\Delta)^{s/2} \tilde{u}_{j_k}|^2 - L^n \int \left(a(Lx) F(\tilde{u}_{j_k}) - \lambda \frac{\tilde{u}_{j_k}^2}{2} \right) \\ &= \frac{L^{n-2s}}{2} \int |(-\Delta)^{s/2} \tilde{u}_{j_k}|^2 - L^n \left(\int G_\infty(\tilde{u}_{j_k}) + o_L(1) \right) < 0, \quad \text{for } L \rightarrow \infty. \end{aligned}$$

Hence, we deduce that

$$c \leq \max_{t \in [0,1]} I(\eta(1, \tilde{u}_{j_k}(\cdot/Lt))) = \max_{t>0} I(\eta(1, \tilde{u}_{j_k}(\cdot/t))) < p - \varepsilon < p,$$

contradicting that fact that $p = c$, provided by Lemma 5.3. By Lemma 5.12, being $\|\tilde{u}_j - u_j\|_{H^s} \rightarrow 0$, we get $I'(u_j) \rightarrow 0$ as $j \rightarrow \infty$. Therefore, $\{u_j\}$ satisfies the assumptions of Corollary 5.8 and since $p = c_\infty$ is not attained by Theorem 1.1, then the splitting lemma holds with $k = 1$, see Corollary 5.8. This yields $u_j(x) = u^1(x - y_j) + o_j(1)$ as $j \rightarrow \infty$ where $y_j \in \mathbb{R}^n$, $|y_j| \rightarrow +\infty$ and u^1 is a solution of the problem at infinity. By making a translation, $u_j(x + y_j) = u^1(x) + o_j(1)$. Applying the barycenter map yields $\beta(u_j(x + y_j)) = \beta(u_j) - y_j = -y_j$ and $\beta(u^1(x) + o_j(1)) = \beta(u^1(x)) + o_j(1)$ by continuity. Then, we reach a contradiction, yielding $b > c_\infty$. \square

Let us consider a positive, radially symmetric, ground state solution $w \in H^s(\mathbb{R}^n)$ to the autonomous problem at infinity. We define the operator $\Pi : \mathbb{R}^n \rightarrow \mathcal{P}$ by

$$\Pi[y](x) := w\left(\frac{x - y}{\vartheta_y}\right),$$

where ϑ_y projects $w(\cdot - y)$ onto \mathcal{P} . Π is continuous as ϑ_y is unique and $\vartheta_y(w(\cdot - y))$ is a continuous function of $w(\cdot - y)$.

Lemma 5.14. $\beta(\Pi[y](x)) = y$ for every $y \in \mathbb{R}^n$.

Proof. Let $v(x) = w((x - y)/\vartheta_y)$, then

$$\mu(v)(x) = \frac{1}{|B_1|} \int_{B_1(x-y)} \left| w\left(\frac{\xi}{\vartheta_y}\right) \right| d\xi = \mu\left(w\left(\frac{\cdot}{\vartheta_y}\right)\right)(x - y),$$

and further, that $\hat{v}(x) = \widehat{w(\cdot/\vartheta_y)}(x - y)$. Using the fact that $\|\hat{v}\|_{L^1} = \|\widehat{w(\cdot/\vartheta_y)}\|_{L^1}$, we get

$$\begin{aligned} \beta(v) &= \frac{1}{\|\hat{v}\|_{L^1}} \int x \widehat{w(\cdot/\vartheta_y)}(x - y) dx \\ &= \frac{1}{\|\hat{v}\|_{L^1}} \int (z + y) \widehat{w(\cdot/\vartheta_y)}(z) dz \\ &= \frac{1}{\|\hat{v}\|_{L^1}} \int z \widehat{w(\cdot/\vartheta_y)}(z) dz + \frac{1}{\|\hat{v}\|_{L^1}} \int y \widehat{w(\cdot/\vartheta_y)}(z) dz \\ &= \beta(w(\cdot/\vartheta_y)) + \frac{y}{\|\hat{v}\|_{L^1}} \int \hat{v}(y + z) dz = y, \end{aligned}$$

since w is radially symmetric. \square

Lemma 5.15. $I(\Pi[y]) \searrow c_\infty$, if $|y| \rightarrow +\infty$.

Proof. Since $\Pi[y] \in \mathcal{P}$, as observed in (4.6), the functional I can be written as

$$I(\Pi[y]) = \frac{s}{n} \int |(-\Delta)^{s/2} w \left(\frac{x-y}{\vartheta_y} \right)|^2 + \frac{1}{n} \int \nabla a(x) \cdot x F \left(w \left(\frac{x-y}{\vartheta_y} \right) \right).$$

Moreover, since $w \in \mathcal{P}_\infty$, by (4.7) we have $I_\infty(w) = \frac{s}{n} \int |(-\Delta)^{s/2} w|^2$ and we obtain

$$\begin{aligned} I(\Pi[y]) &= \frac{s\vartheta_y^{n-2s}}{n} \int |(-\Delta)^{s/2} w|^2 \\ &\quad + \frac{\vartheta_y^n}{n} \int \nabla a(\vartheta_y x + y) \cdot (\vartheta_y x + y) F(w) \\ &= \vartheta_y^{n-2s} I_\infty(w) + \frac{\vartheta_y^n}{n} \int \nabla a(\vartheta_y x + y) \cdot (\vartheta_y x + y) F(w) \quad (> c_\infty). \end{aligned}$$

By Lebesgue Dominated Convergence Theorem, (1.7) and $\vartheta_y \rightarrow 1$ if $|y| \rightarrow +\infty$, we get

$$\lim_{|y| \rightarrow \infty} \int \nabla a(\vartheta_y x + y) \cdot (\vartheta_y x + y) F(w) = 0.$$

Therefore, $I(\Pi[y]) \searrow c_\infty$ if $|y| \rightarrow +\infty$ and the proof is complete. \square

Lemma 5.16. *Let C be a positive constant such that $|F(s)| \leq Cs^2$. Assume*

$$(A6) \quad \|a_\infty - a\|_{L^\infty} < \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\widehat{\vartheta}^n \|w\|_2^2 C}, \quad \widehat{\vartheta} = \sup_{y \in \mathbb{R}^n} \vartheta_y.$$

Then $I(\Pi[y]) < \min\{c_\sharp, 2c_\infty\}$ for every $y \in \mathbb{R}^n$.

Proof. The maximum of $t \mapsto I_\infty(w(\cdot/t))$ is attained at $t = 1$. Since $\vartheta_y > 1$, using (A6), we obtain

$$\begin{aligned} I(\Pi[y]) &= I_\infty(\Pi[y]) + I(\Pi[y]) - I_\infty(\Pi[y]) \leq I_\infty(w) + \int (a_\infty - a(x)) F(\Pi[y]) \\ &< c_\infty + \frac{\min\{c_\sharp, 2c_\infty\} - c_\infty}{\widehat{\vartheta}^n \|w\|_2^2 C} \int C w^2 \left(\frac{x-y}{\vartheta_y} \right) \\ &= c_\infty + \frac{(\min\{c_\sharp, 2c_\infty\} - c_\infty) \vartheta_y^n}{\widehat{\vartheta}^n \|w\|_2^2} \|w\|_2^2 = \min\{c_\sharp, 2c_\infty\}, \end{aligned}$$

which concludes the proof. \square

Remark 5.17. Replacing (A6) with $\|a_\infty - a\|_{L^\infty} < c_\infty \widehat{\vartheta}^{-n} \|w\|_2^{-2} C^{-1}$, one gets $I(\Pi[y]) < 2c_\infty$.

We will need a version of the Linking Theorem with Cerami condition by [4, Theorem 2.3].

Definition 5.18. Let S be a closed subset of a Banach space X and Q a sub manifold of X with relative boundary ∂Q . We say that S and ∂Q link if the following facts hold

- 1) $S \cap \partial Q = \emptyset$;
- 2) for any $h \in C^0(X, X)$ with $h|_{\partial Q} = id$, then $h(Q) \cap S \neq \emptyset$.

Moreover, if S and Q are as above and B is a subset of $C^0(X, X)$, then S and ∂Q link with respect to B if 1) and 2) hold for any $h \in B$.

Theorem 5.19. *Suppose that $I \in C^1(X, \mathbb{R})$ is a functional satisfying (Ce) condition. Consider a closed subset $S \subset X$ and a submanifold $Q \subset X$ with relative boundary ∂Q such that*

- a) S and ∂Q link;
- b) $\alpha = \inf_{u \in S} I(u) > \sup_{u \in \partial Q} I(u) = \alpha_0$.
- c) $\sup_{u \in Q} I(u) < +\infty$.

If $B = \{h \in C^0(X, X) : h|_{\partial Q} = \text{id}\}$, then $\tau = \inf_{h \in B} \sup_{u \in Q} I(h(u)) \geq \alpha$ is a critical value of I .

Proof of Theorem 1.2 concluded. We follow the argument in [1, Theorem 7.7]. Since we have $b > c_\infty$ from Lemma 5.13 and $I(\Pi[y]) \searrow c_\infty$ if $|y| \rightarrow \infty$ from Lemma 5.15, there exists $\bar{\rho} > 0$ such that

$$(5.11) \quad c_\infty < \max_{|y|=\bar{\rho}} I(\Pi[y]) < b.$$

In order to apply the linking theorem, we take

$$Q := \Pi(\overline{B_{\bar{\rho}}(0)}), \quad S := \{u \in H^s(\mathbb{R}^n) : u \in \mathcal{P}, \beta(u) = 0\},$$

and we show that ∂Q and S link with respect to $\mathcal{H} = \{h \in C(Q, \mathcal{P}) : h|_{\partial Q} = \text{id}\}$. Since $\beta(\Pi[y]) = y$ from Lemma 5.14, we have that $\partial Q \cap S = \emptyset$, as if $u \in S$, then $\beta(u) = 0$, and if $u \in \partial Q$, $u = \Pi[y]$ for some $y \in \mathbb{R}^n$ with $|y| = \bar{\rho}$ and then $\beta(u) = y \neq 0$. Now we show that $h(Q) \cap S \neq \emptyset$ for any $h \in \mathcal{H}$. Given $h \in \mathcal{H}$, let $T : \overline{B_{\bar{\rho}}(0)} \rightarrow \mathbb{R}^n$ by defined by $T(y) = \beta \circ h \circ \Pi[y]$. The function T is continuous, by composition. Moreover, for $|y| = \bar{\rho}$, we have that $\Pi[y] \in \partial Q$, thus $h \circ \Pi[y] = \Pi[y]$, as $h|_{\partial Q} = \text{id}$, and hence $T(y) = y$ by Lemma 5.14. By Brower Fixed Point Theorem there is $\tilde{y} \in B_{\bar{\rho}}(0)$ with $T(\tilde{y}) = 0$, which implies $h(\Pi[\tilde{y}]) \in S$. Then $h(Q) \cap S \neq \emptyset$ and S and ∂Q link. Now, from (5.11), we may write

$$b = \inf_S I > \max_{\partial Q} I$$

Let us define

$$d = \inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)).$$

It is $d \geq b$. In fact, if $h \in \mathcal{H}$, there exists $w \in S$ with $w = h(v)$ for some $v \in \Pi(\overline{B_{\bar{\rho}}(0)})$. Therefore,

$$\max_{u \in Q} I(h(u)) \geq I(h(v)) = I(w) \geq \inf_{u \in S} I(u) = b,$$

and hence $d \geq b$, which implies $d > c_\infty$. Furthermore, if $h = \text{id}$, then

$$\inf_{h \in \mathcal{H}} \max_{u \in Q} I(h(u)) < \max_{u \in Q} I(u) < \min\{c_\sharp, 2c_\infty\},$$

in light of Lemma 5.16. Then $d \in (c_\infty, \min\{c_\sharp, 2c_\infty\})$ and thus from Lemma 5.9 the (Ce) condition is satisfied at level d . Then, by the linking theorem, d is a critical level for I . \square

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